Formulari de Mètodes Matemàtics II

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EQUACIONS DIFERENCIALS

$$u(x,t) = X(x)T(t)$$

Coef. constants:

$$a_2 y''(x) + a_1 y'(x) + a_0 y(x) = 0 \quad \rightarrow \quad y(x) \sim e^{\alpha x}$$

$$\alpha_1 \neq \alpha_2 \quad \rightarrow \quad y(x) = Ae^{\alpha_1 x} + Be^{\alpha_2 x}$$

$$\alpha_1 = \alpha_2 \quad \rightarrow \quad y(x) = Ae^{\alpha x} + Bxe^{\alpha x}$$

Segona solució: $y''(x) + p(x)y'(x) + q(x)y(x) = 0$

$$y_2(x) = y_1(x) \cdot u(x) \quad \rightarrow \quad y_2(x) = y_1(x) \int \frac{e^{-\int p(x)dx}}{y_1^2(x)} dx$$

Coef. no contstants: *Equació de Cauchy-Euler*

$$ax^2 y''(x) + bxy'(x) + cy(x) = 0 \quad \rightarrow \quad y(x) \sim x^\lambda, \; 0 < x < \infty$$

$$\lambda_1 \neq \lambda_2 \quad \rightarrow \quad y(x) = Ax^{\lambda_1} + Bx^{\lambda_2}$$

$$\lambda_1 = \lambda_2 \quad \rightarrow \quad y(x) = Ax^\lambda + Bx^\lambda \ln x$$

$$\lambda_n = \alpha \pm i\beta \quad \rightarrow \quad y(x) = x^\lambda [k_1 \cos(\beta \ln x) - k_2 \sin(\beta \ln x)]$$

Mètode Fröbenius

$$w''(z) + p(z)w'(z) + q(z)w(z)$$

i. Si z_0 és punt ordinari:

$$w(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

ii. Si z_0 és punt singular regular:

$$w(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^{(n+\alpha)}$$

FUNCIONS ESPECIALS

Gamma d'Euler:

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \quad x > 0$$

Funcions de Bessel:

$$x^2 y''(x) + xy'(x) + (x^2 - \nu^2)y(x) = 0$$

$$a_0 = \frac{1}{2^\nu \Gamma(\nu+1)} \qquad J_\nu = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\Gamma(\nu+n+1)} \left(\frac{x}{2}\right)^{\nu+2n}$$

$$\nu \notin \mathbb{N} \qquad y(x) = AJ_\nu(x) + BJ_{-\nu}(x)$$

$$\nu \in \mathbb{N} \qquad y(x) = AJ_\nu(x) + BY_\nu(x), \quad \text{també } \forall \nu$$

Modif: $\quad x^2 y''(x) + xy'(x) + (\lambda x^2 - \nu^2)y(x) = 0$

$$y(x) = AJ_\nu(\lambda x) + BY_\nu(\lambda x)$$

$$\lambda = \pm i \quad \rightarrow \quad y(x) = AI_\nu(x) + BK_\nu(x)$$

Polinomis de Legendre:

$$w''(z) - \frac{2z}{1-z^2} w'(z) + \frac{\nu(\nu+1)}{1-z^2} w(z) = 0$$

$$\nu = n \quad \rightarrow \quad w(z) = AP_n(z) + BQ_n(z)$$

$$\nu \neq n \quad \rightarrow \quad a_{r+2} = \frac{(r+1) - \nu(\nu+1)}{(r+1)(r+2)} \quad \nu \text{ parell: pol.Leg.}$$

STÜRM-LIOUVILLE

$$[p(x)y'(x)]' + q(x)y(x) + \lambda w(x)y(x) = 0$$

$$\begin{cases} \alpha y(a) + \beta y'(a) = 0 \\ \gamma y(b) + \delta y'(b) = 0 \end{cases}$$

λ_n : val.prop. $\qquad \varphi_n$: func.prop. $\qquad \langle \varphi_n | \varphi_m \rangle \propto \delta_{mn}$

Producte escalar:

$$\langle f(x) | g(x) \rangle = \int_a^b f^*(x) g(x) w(x) dx$$

SL periòdic: $y(a) = y(b);\; y'(a) = y'(b)$

SL singular: $p(a) = 0 \quad \rightarrow \quad y$ acotada en a.

TRANSFORMADA DE LAPLACE

$$\mathcal{L}[f(t)] = F(s) = \int_0^\infty f(t) e^{-st} dt \qquad s > 0$$

Propietats:

$$\mathcal{L}[e^{at} f(t)] = F(s-a) \quad \mathcal{L}[f(at)] = \frac{1}{a} F\left(\frac{s}{a}\right)$$

$$\mathcal{L}[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} F(s) \quad \mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}$$

$$\mathcal{L}[f^{n)}(t)] = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \ldots - s f^{n-2)}(0) - f^{n-1)}(0)$$

Convolució:

$$(f * g)(t) = \int_0^t f(t-\xi) g(\xi) d\xi$$

$$\mathcal{L}[(f*g)(t)] = F(s)G(s) \qquad \mathcal{L}^{-1}[F(s)G(s)] = (f*g)(t)$$

SÈRIES DE FOURIER

Període: L

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{2\pi nx}{L}\right) + b_n \sin\left(\frac{2\pi nx}{L}\right)$$

$$a_n = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \cos\left(\frac{2\pi nx}{L}\right) dx$$

$$b_n = \frac{2}{L} \int_{x_0}^{x_0+L} f(x) \sin\left(\frac{2\pi nx}{L}\right) dx$$

Parseval:

$$\frac{1}{L} \int_{x_0}^{x_0+L} |f(x)|^2 dx = \left(\frac{1}{2} a_0\right)^2 + \frac{1}{2} \sum_{n=1}^{\infty} (a_n^2 + b_n^2)$$

TRANSFORMADA DE FOURIER

$$\mathcal{F}[f(t)] = \hat{f}(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\mathrm{d}t$$

$$f(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(\omega)\mathrm{e}^{i\omega t}\mathrm{d}\omega$$

Propietats

$$\mathcal{F}[f(at)] = \frac{1}{a}\hat{f}\left(\frac{\omega}{a}\right) \qquad \mathcal{F}[f(t+a)] = \mathrm{e}^{ia\omega}\hat{f}(\omega)$$

$$\mathcal{F}[\mathrm{e}^{\alpha t}f(t)] = \hat{f}(\omega + i\alpha) \qquad \mathcal{F}[f^{n)}(t)] = (-i\omega)^n\hat{f}(\omega)$$

Convolució:

$$(f * g)(t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(\xi)g(t-\xi)\mathrm{d}\xi$$

$$\mathcal{F}[(f*g)(t)] = \hat{f}(\omega)\hat{g}(\omega) \qquad \mathcal{F}^{-1}[\hat{f}(\omega)\hat{g}(\omega)] = (f*g)(t)$$

FRACCIONS PARCIALS

$$F(s) = \frac{P(s)}{Q(s)}$$

Grau numerador $\geq$ grau denominador: divisió de polinomis.
Garu numerador$<$ grau denominador: fraccions parcials.

1. Per a cada factor de la forma $as + b$ de $Q(s)$:

$$\frac{A}{as+b}$$

2. Per a cada factor lineal repetit de la forma $(as+b)^n$:

$$\frac{A_1}{as+b} + \frac{A_2}{(as+b)^2} + \ldots + \frac{A_n}{(as+b)^n}$$

3. Per a cada factor quadràtic de la forma $as^2 + bs + c$:

$$\frac{As+B}{as^2+bs+c}$$

4. Per a cada factor quadràtic repetit de la forma $(as^2 + bs + c)^n$:

$$\frac{A_1s+B_1}{as^2+bs+c} + \frac{A_2s+B_2}{(as^2+bs+c)^2} + \ldots + \frac{A_ns+B_n}{(as^2+bs+c)^n}$$

Pols simples (transformada de Laplace):

$$F(s) = \frac{P(s)}{(s-\alpha_1)(s-\alpha_2)\cdot\ldots\cdot(s-\alpha_n)}$$

$$f(t) = \mathcal{L}^{-1}(F(s)) = \sum_{i=1}^{n} \lim_{s\to\alpha_i}(s-\alpha_i)F(s)\mathrm{e}^{\alpha_i t}$$