Formulari de Relativitat General

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I. Geometria Diferencial

Camps de vectors i corbes integrals

$$\vec{v} \equiv \frac{\mathrm{d}}{\mathrm{d}\lambda} = \left.\frac{\mathrm{d}x^i}{\mathrm{d}\lambda}\right|_p \frac{\partial}{\partial x^i} = v^i(x)\partial_i \quad \to \quad v^i(x) = \frac{\mathrm{d}x^i}{\mathrm{d}\lambda}$$

Commutador: $[\vec{v},\vec{w}] = \left(v^i\partial_i w^j - w^i\partial_i v^j\right)\partial_j = -[\vec{w},\vec{v}]$

$\{\vec{v}_i\}$ és base coordenada $\Leftrightarrow [\vec{v}_{(j)},\vec{v}_{(k)}] = 0$.

Tensors en una varietat

Vectors i vectors duals:

$$\vec{v} = v^i\vec{e}_i = v^i\partial_i \in T_p \quad \tilde{p} = p_i\tilde{\omega}^i = p_i\tilde{\mathrm{d}}x^i \in T_p^*$$

$$\tilde{\omega}^j(\vec{e}_i) = \tilde{\mathrm{d}}x^j(\partial_i) = \frac{\partial x^i}{\partial x^j} = \delta^i_j \quad \tilde{\mathrm{d}}f = \frac{\partial f}{\partial x^i}\tilde{\mathrm{d}}x^i$$

$$\vec{e}_{i'} = \Lambda^i_{i'}\vec{e}_i \;\; \tilde{\omega}^{i'} = \Lambda^{i'}_i\tilde{\omega}^i \;\; v^{i'} = \Lambda^{i'}_i v^i \;\; p_{i'} = \Lambda^i_{i'}p_i$$

Tensor $\binom{N}{M}$: $\mathbf{T} : T_p^* \times \ldots \times T_p^* \times T_p \times \ldots \times T_p \to \mathbb{R}$

$$\mathbf{T} = T^{i\ldots j}_{k\ldots l}\left(\vec{e}_i \otimes \ldots \otimes \vec{e}_j \otimes \tilde{\omega}_k \otimes \ldots \otimes \tilde{\omega}_l\right)$$

$$T^{i'\ldots j'}_{k'\ldots l'} = \Lambda^{i'}_i \ldots \Lambda^{j'}_j \Lambda^k_{k'} \ldots \Lambda^l_{l'} T^{i\ldots j}_{k\ldots l}$$

Mètrica: $\mathbf{g}$, tensor $\binom{0}{2}$:

$$\mathbf{g} = g_{ij}\tilde{\omega}^i \otimes \tilde{\omega}^j = \mathbf{g}(\vec{e}_i,\vec{e}_j)(\tilde{\omega}^i \otimes \tilde{\omega}^j) \equiv \vec{e}_i \cdot \vec{e}_j(\tilde{\omega}^i \otimes \tilde{\omega}^j)$$

$$g_{ij}g^{ik} = \delta^k_j \quad v_i = g_{ij}v^j \quad v^i = g^{ij}v_j \quad \vec{v}\cdot\vec{w} = g_{ij}v^iw^j = v_jw^j$$

Derivada de Lie

$$\pounds_{\vec{v}}f|_{t_0} = v^i\frac{\partial f}{\partial x^i} = \vec{v}(f) \quad \pounds_{\vec{u}}\vec{v} = [\vec{u},\vec{v}]$$

Transport d'un vector per Lie:

$$\pounds_{\vec{u}}\vec{v} = 0 \quad \to \quad u^i\partial_i v^j - v^i\partial_i u^j = 0$$

Propietats:

$$\pounds_{\vec{v}}\left(\tilde{p}(\vec{u})\right) = \left(\pounds_{\vec{v}}\tilde{p}\right)(\vec{u}) + \tilde{p}\left(\pounds_{\vec{v}}\vec{u}\right)$$

$$\pounds_{\vec{v}}\left(\mathbf{T}\otimes\mathbf{Q}\right) = \left(\pounds_{\vec{v}}\mathbf{T}\right)\otimes\mathbf{Q} + \mathbf{T}\otimes\left(\pounds_{\vec{v}}\mathbf{Q}\right)$$

$$\pounds_{\vec{v}}(\tilde{\mathrm{d}}f) = \tilde{\mathrm{d}}\pounds_{\vec{v}}(f) \quad \pounds_{a\vec{v}} = a\pounds_{\vec{v}} \quad \pounds_{[\vec{v},\vec{u}]} = [\pounds_{\vec{v}},\pounds_{\vec{u}}]$$

Simetries i vectors de Killing

Equació de Killing:

$$\pounds_{\vec{v}}\mathbf{g} = 0$$

$$(\pounds_{\vec{v}}\mathbf{g})_{ij} = v^k\partial_k g_{ij} + \partial_i v^k g_{kj} + \partial_j v^k g_{ik}$$

Nombre màxim: $\frac{n(n+1)}{2}$ $\quad \partial_j v_i + \partial_i v_j = 2v_l\Gamma^l_{ji}$

p-formes

Components, dimensió i producte exterior:

$$\alpha_{i\ldots j} = \alpha_{[i\ldots j]} \qquad C^n_p = \frac{n!}{p!(n-p)!} \qquad \tilde{\omega}\wedge\tilde{\sigma} = \tilde{\omega}\otimes\tilde{\sigma} - \tilde{\sigma}\otimes\tilde{\omega}$$

En una base:

$$\tilde{\alpha} = \frac{1}{p!}\alpha_{i\ldots j}\tilde{\omega}^i\wedge\ldots\wedge\tilde{\omega}^j$$

$$(\tilde{p}\wedge\tilde{q}) = \frac{1}{(p+q)!}(\tilde{p}\wedge\tilde{q})_{i\ldots l}\tilde{\omega}^i\wedge\ldots\wedge\tilde{\omega}^l = \frac{1}{p!q!}p_{[i\ldots j}q_{k\ldots l]}\tilde{\omega}^i\wedge\ldots\wedge\tilde{\omega}^l$$

Producte interior $i(x):\ \Lambda^p \to \Lambda^{p-1}$

$$\tilde{\alpha}(\vec{x}) = \frac{1}{(p-1)!}\alpha_{ij\ldots l}x^i\tilde{\omega}^j\wedge\ldots\wedge\tilde{\omega}^l$$

Dual de Hodge $*:\Lambda^m \to \Lambda_{n-m}$

$$({}^*T)_{j\ldots l} = \frac{1}{m!}\omega_{i\ldots kj\ldots l}T^{i\ldots k} \quad ({}^*B)^{j\ldots l} = \frac{1}{m!}\omega^{i\ldots kj\ldots l}B_{j\ldots k}$$

$n-$formes i símbol de Levi Civita

$$\omega_{1\ldots n}\omega^{1\ldots n} = 1 \quad \omega_{i\ldots j}\omega^{i\ldots j} = n! \quad \omega_{i\ldots j} = f\epsilon_{i\ldots j} \quad \omega^{i\ldots j} = \frac{1}{f}\epsilon^{i\ldots j}$$

$$\epsilon^{ijk}\epsilon_{imn} = \delta^j_m\delta^k_n - \delta^k_n\delta^j_m$$

$$\epsilon^{a_1\ldots a_j a_{j+1}\ldots a_n}\epsilon_{a_1\ldots a_j b_{j+1}\ldots b_n} = (n-j)!j!\delta^{[a_{j+1}}_{b_{j+1}}\ldots\ \delta^{a_n]}_{b_n}$$

Determinant i canvi de base:

$$\det(A) = \epsilon_{ij\ldots k}A^{1i}A^{2j}\ldots A^{nk} = \frac{1}{n!}\epsilon_{ab\ldots c}\epsilon_{ij\ldots l}A^{ai}A^{bj}\ldots A^{nk}$$

$$f' = f\det(\Lambda)$$

Forma volum mètric:

$$\mathbf{g} = \pm\tilde{\omega}^{1'}\otimes\tilde{\omega}^{1'}\pm\ldots\pm\tilde{\omega}^{n'}\otimes\tilde{\omega}^{n'}.$$

$$\tilde{\omega} = \tilde{\omega}^{1'}\wedge\ldots\wedge\tilde{\omega}^{n'} = \sqrt{|\det(\mathbf{g})|}\tilde{\mathrm{d}}x^1\wedge\ldots\wedge\tilde{\mathrm{d}}x^n$$

Derivada exterior

$$\tilde{\mathrm{d}}\tilde{\alpha} = \frac{1}{p!}\partial_l\alpha_{i\ldots j}\tilde{\mathrm{d}}x^l\wedge\tilde{\mathrm{d}}x^i\wedge\ldots\wedge\tilde{\mathrm{d}}x^j$$

$$\tilde{\mathrm{d}}(\tilde{\alpha}+\tilde{\beta}) = \tilde{\mathrm{d}}\tilde{\alpha}+\tilde{\mathrm{d}}\tilde{\beta} \quad \tilde{\mathrm{d}}(\tilde{\alpha}\wedge\tilde{\beta}) = (\tilde{\mathrm{d}}\tilde{\alpha})\wedge\tilde{\beta} + (-1)^p\tilde{\alpha}\wedge(\tilde{\mathrm{d}}\tilde{\beta})$$

$$\tilde{\mathrm{d}}(\tilde{\mathrm{d}}\tilde{\alpha}) = 0 \quad \tilde{\mathrm{d}}(f) = \tilde{\mathrm{d}}f = \mathrm{grad}f \quad \tilde{\mathrm{d}}(f\tilde{\mathrm{d}}g) = \tilde{\mathrm{d}}f\wedge\tilde{\mathrm{d}}g$$

Aplicacions:

$$\mathbb{E}^3 \;\to\; {}^*(\tilde{u}\wedge\tilde{v}) \equiv \vec{u}\times\vec{v}$$

$${}^*\tilde{\mathrm{d}}\tilde{a} \equiv \vec{\nabla}\times\vec{a} \qquad \tilde{\mathrm{d}}{}^*\vec{a} = (\mathrm{div}\vec{a})\tilde{\omega} = (\mathrm{div}\vec{a})\tilde{\mathrm{d}}x^1\wedge\tilde{\mathrm{d}}x^2\wedge\mathrm{d}x^3$$

Integració

$$\int_\Omega\tilde{\alpha} = \int_{\mathcal{U}\subset\mathbb{R}^n}\tilde{\alpha}(\vec{e}_i,\ldots,\vec{e}_j)\frac{\partial x^i}{\partial\lambda^{\bar{a}}}\cdots\frac{\partial x^j}{\partial\lambda^{\bar{b}}}\tilde{\mathrm{d}}x^{\bar{a}}\ldots\tilde{\mathrm{d}}\lambda^{\bar{b}}$$

Teorema de Stokes i 0-formes:

$$\int_\Omega\tilde{\mathrm{d}}\tilde{\alpha} = \int_{\partial\Omega}\tilde{\alpha} \qquad \int_\Omega{}^*f = \int_\Omega f\sqrt{|\det(g)|}\tilde{\mathrm{d}}x^1\wedge\ldots\wedge\tilde{\mathrm{d}}x^n$$

Teorema de Gauss:

$$\int_{\partial\Omega}{}^*\vec{v} = \int_\Omega\mathrm{d}^*\vec{v} \qquad \int_{\partial\Omega}v^in_i\tilde{\mathrm{d}}\lambda^{\bar{1}}\wedge\ldots\wedge\tilde{\mathrm{d}}\lambda^{\overline{n-1}} = \int_\Omega(\mathrm{div}_{\tilde{\omega}}\vec{v})\tilde{\omega}$$

Connexió i derivada covariant

$$\nabla_{\vec{x}}(\vec{y}\otimes\vec{z}) = (\nabla_{\vec{x}}\vec{y})\otimes\vec{z} + \vec{y}\otimes(\nabla_{\vec{x}}\vec{z})$$
$$\nabla_{\vec{x}}(\tilde{\sigma}(\vec{y})) = (\nabla_{\vec{x}}\tilde{\sigma})(\vec{y}) + \tilde{\sigma}(\nabla_{\vec{x}}\vec{y})$$
$$\nabla_{\vec{x}}(f\vec{z}) = (\nabla_{\vec{x}}f)\vec{z} + f(\nabla_{\vec{x}}\vec{z})$$
$$\nabla_{\vec{x}}(t) = \vec{x}(t)$$
$$\nabla_{f\vec{x}+g\vec{y}}\vec{z} = f\nabla_{\vec{x}}\vec{z} + g\nabla_{\vec{y}}\vec{z}$$

$$\nabla_{\vec{e}_j}\vec{e}_k = \Gamma^i_{kj}\vec{e}_i \quad (\nabla\vec{y})(\tilde{\sigma},\vec{x}) = \nabla_j y^i\sigma_i x^j \;\to\; x^j\nabla_j y^i = (\nabla_{\vec{x}}\vec{y})^i$$

$$\nabla_j y^i = \partial_j(y^i) + \Gamma^i_{kj}y^k \quad \nabla_j\sigma_i = \partial_j\sigma_i - \Gamma^k_{ij}\sigma_k$$

$$\nabla_i T^{j\cdots k}_{l\cdots m} = \partial_i T^{j\cdots k}_{l\cdots m} + \Gamma^j_{si}T^{s\cdots k}_{l\cdots m} + \ldots + \Gamma^k_{si}T^{j\cdots s}_{l\cdots m} - \Gamma^s_{li}T^{j\cdots k}_{s\cdots m} - \ldots - \Gamma^s_{mi}T^{j\cdots k}_{l\cdots s}$$

Connexió simètrica:

$$\Gamma^i_{kl} = \Gamma^i_{lk}$$

$$\nabla_{\vec{y}}\vec{x} - \nabla_{\vec{x}}\vec{y} = [\vec{x}, \vec{y}] = \pounds_{\vec{x}}\vec{y}$$

$$\pounds_{\vec{x}}\tilde{\sigma} = x^k \nabla_k \sigma_i + \sigma_k \nabla_i x^k$$

Connexió mètrica:

$$\nabla \mathbf{g} = 0 \qquad \nabla_\alpha g_{\mu\nu} = 0$$

$$\Gamma^l_{ji} = \frac{1}{2} g^{lm}(\partial_j g_{mi} + \partial_i g_{mj} - \partial_m g_{ij})$$

Transport paral·lel:

$$\nabla_{\vec{v}}\vec{u} = 0 \;\rightarrow\; v^k \partial_k u^i + \Gamma^i_{lk} v^l u^k = 0$$

$$\frac{\mathrm{d}u^i}{\mathrm{d}\lambda} + \Gamma^i_{lk} \frac{\mathrm{d}x^l}{\mathrm{d}\lambda} u^k = 0$$

Geodèsiques

$$\nabla_{\vec{v}}\vec{v} = 0 \;\rightarrow\; \frac{\mathrm{d}^2 x^k}{\mathrm{d}\lambda^2} + \Gamma^k_{li} \frac{\mathrm{d}x^l}{\mathrm{d}\lambda} \frac{\mathrm{d}x^i}{\mathrm{d}\lambda} = 0$$

Curvatura de la connexió

Tensor de Riemann: $\mathbb{R}(\vec{x}, \vec{y}) \equiv [\nabla_{\vec{x}}, \nabla_{\vec{y}}] - \nabla_{[\vec{x},\vec{y}]}$

$$[\nabla_{\vec{e}_i}, \nabla_{\vec{e}_j}]\vec{e}_k - \nabla_{[\vec{e}_i,\vec{e}_j]}\vec{e}_k = R^l_{kij}\vec{e}_l$$

$$R^l_{kij} = \partial_i \Gamma^l_{kj} - \partial_j \Gamma^l_{ki} + \Gamma^m_{kj}\Gamma^l_{mi} - \Gamma^m_{ki}\Gamma^l_{mj}$$

Amb $\frac{n^2(n^2-1)}{12}$ components independents.

$$R^l_{k(ij)} = 0 \qquad R^l_{[kij]} = 0 \qquad R^l_{k[ij;m]} = 0$$

$$R_{klmn} = g_{ki} R^i_{lmn}$$

Tensor de Ricci i escalar de Ricci:

$$R_{ik} = R^m_{imk} \qquad R = g^{ik} R_{ik} \quad \nabla^\mu R_{\rho\mu} = -\frac{1}{2}\nabla_\rho R$$

Tensor d'Einstein:

$$G_{ij} \equiv R_{ij} - \frac{1}{2} g_{ij} R \qquad G = -R \qquad \nabla_i G^{ij} = 0$$

Divergència (respecte connexió Levi-Civita):

$$\Gamma^\mu_{\mu\lambda} = \frac{1}{\sqrt{|\mathbf{g}|}} \partial_\lambda \sqrt{|\mathbf{g}|} \quad \nabla_\mu v^\mu = \partial_\mu v^\mu + \Gamma^\mu_{\mu\lambda} v^\lambda = \frac{1}{\sqrt{|\mathbf{g}|}} \partial_\mu \left(\sqrt{|\mathbf{g}|} v^\mu \right)$$

II. Relativitat General

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} = 8\pi T_{\mu\nu}$$

Simetria Esfèrica i Forats Negres

Estàtic: killing temps + $g_{0i} = 0$

$$\mathrm{d}s^2 = -\mathrm{e}^{2\phi(t,r)}\mathrm{d}t^2 + \mathrm{e}^{2\Lambda(t,r)}\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

$$l_{12} = \int_{r_1}^{r_2} \sqrt{\mathrm{e}^{2\Lambda}} \mathrm{d}r \qquad \frac{\omega_1}{\omega_2} = \mathrm{e}^{\phi(r_2) - \phi(r_1)}$$

Mètrica de Schwarzschild:

$$\mathrm{d}s^2 = -\left(1 - \frac{2M}{r}\right)\mathrm{d}t^2 + \frac{\mathrm{d}r^2}{1 - \frac{2M}{r}} + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\phi^2)$$

Dinàmica:

$$g_{\alpha\beta} u^\alpha u^\beta = -1 \quad g_{\alpha\beta} k^\alpha k^\beta = 0 \quad -k_\alpha u^\alpha = \omega$$

Si ξ^α és Killing i v^μ geodèsica, $\xi^\alpha v_\alpha = g_{\alpha\beta}\xi^\alpha v^\beta = \text{cnst.}$

$$u_0 \equiv -\hat{E} \quad k_0 \equiv -E \quad u_\phi \equiv \hat{L} \quad k_\phi \equiv L$$

Trajectòries:

$$\left(\frac{\mathrm{d}r}{\mathrm{d}\tau}\right)^2 = \hat{E}^2 - \left(1 - \frac{2M}{r}\right)\left(1 + \frac{\hat{L}^2}{r^2}\right) \qquad \left(\frac{\mathrm{d}r}{\mathrm{d}\lambda}\right)^2 = E^2 - \left(1 - \frac{2M}{r}\right)\frac{L^2}{r^2}$$

Mínim estable:

$$\hat{L}^2 = \frac{Mr}{1 - \frac{3M}{r}} \quad \hat{E}^2 = \frac{\left(1 - \frac{2M}{r}\right)^2}{1 - \frac{3M}{r}}$$

$$r = \frac{\hat{L}^2}{2M}\left(1 \pm \sqrt{1 - \frac{12M^2}{\hat{L}^2}}\right)$$

Coordenades de Kruskal-Szekeres:

$$X = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \mathrm{e}^{r/4M} \cosh\left(\frac{t}{4M}\right)$$

$$T = \left(\frac{r}{2M} - 1\right)^{\frac{1}{2}} \mathrm{e}^{r/4M} \sinh\left(\frac{t}{4M}\right)$$

$$\mathrm{d}s^2 = \frac{32M^3 \mathrm{e}^{-r/2M}}{r}(-\mathrm{d}T^2 + \mathrm{d}X^2) + r^2 \mathrm{d}\Omega^2$$

Teoria Linealitzada i Ones Gravitacionals

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad |h_{\mu\nu}| \ll 1 \qquad \bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h^\alpha_\alpha$$

$$h_{\mu\nu} \;\rightarrow\; h'_{\mu\nu} = h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)}$$

Gauge de Lorentz: $\partial^\mu \bar{h}_{\mu\nu} = 0 \quad \Box^2 \xi_\nu = \partial^\mu h_{\mu\nu}$

Equacions linealitzades + gauge: $\partial_\alpha \partial^\alpha \bar{h}_{\mu\nu} = -2\kappa T_{\mu\nu}$

Gauge TT: $h_{\alpha 0} = 0$, $h^\alpha_\alpha = 0$, $\partial^i h_{ik} = 0$

$$\bar{h}_{\mu\nu} \;\rightarrow\; \bar{h}'_{\mu\nu} = h_{\mu\nu} - 2\partial_{(\mu}\xi_{\nu)} + \partial^\alpha \xi_\alpha$$

Ones planes:

$$\bar{h}_{\mu\nu} = \mathrm{Re}\left[A_{\mu\nu} \exp(i l_\alpha x^\alpha)\right] \qquad l^\alpha = (\omega, \vec{k})$$

$$l_\alpha l^\alpha = \bar{h}_{\mu\nu} l^\nu = R_{\alpha\beta\mu\nu} l^\nu = l^\alpha \nabla_\alpha l^\mu = \nabla_\alpha l^\mu = 0 \qquad i l_\nu = \partial_\nu$$

Potència:

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} = \frac{G}{5c^5}\langle \dddot{I}_{ij} \dddot{I}_{ij} \rangle \quad \dddot{I}_{ij} = D_{ij} - \frac{1}{3}\delta_{ij} D^k_k$$

$$D^{ij} = \int T^{00}(x) x^{i'} x^{j'} \mathrm{d}^3 x' = \sum_\alpha m_\alpha x^i_\alpha x^j_\alpha$$

Límit Newtonià:

Schw: $\mathrm{d}s^2 = -(1 - 2\phi)\mathrm{d}t^2 + \mathrm{d}x^2 + \mathrm{d}y^2 + \mathrm{d}z^2 \qquad \phi = \frac{M}{R+x}$

$$p \ll \rho \qquad \mathrm{d}t = \mathrm{d}\tau \qquad |\partial_0| \ll |\partial_i|$$

$$u^\mu = (1, v^i) \quad |v^i| \ll 1$$